

Exponential Lévy-type models with stochastic volatility and stochastic jump intensity

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Abstract

We consider the problem of valuing a European option written on an asset whose dynamics are described by an exponential Lévy-type model. Both the volatility and jump-intensity are allowed to vary stochastically in time through common driving factors – one fast-varying and one slow-varying. Our results extend the class of multiscale stochastic volatility models of Fouque, Papanicolaou, Sircar, and Solna (2011) to models of the exponential Lévy type. Using generalized Fourier transform techniques we derive an explicit formula for the approximate price of a European-style derivative. For derivatives with smooth and bounded payoffs we establish the accuracy of our pricing approximation. As an example of our framework, we extend the jump-diffusion model of Merton (1976) to include stochastic volatility and stochastic jump-intensity. We perform a calibration to S&P500 options and find that the extended Merton model with a single fast-varying factor of volatility provides a better fit to implied volatility than both the traditional Merton model and the fast mean-reverting stochastic volatility models of Fouque, Papanicolaou, Sircar, and Solna (2011).

Key words: Lévy-type process, stochastic volatility, stochastic jump-intensity.

1 Introduction

An *exponential Lévy model* is an equity model in which an underlying $S = e^X$ is described by the exponential of a Lévy process X . Such models extend the geometric Brownian motion description of Black and Scholes (1973) by allowing the underlying S to experience jumps, the need for which is well-documented in literature (see, Eraker (2004) and references therein). In addition to allowing the underlying S to jump, exponential

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Lévy models are important because they capture many of the stylized features of asset prices, such as heavy tails, high-kurtosis and asymmetry of log returns.

Several well-known models fit within the exponential Lévy class: the jump-diffusion model of Merton (1976), the pure jump models of Mandelbrot (1963), the variance gamma model of Madan, Carr, and Chang (1998), the extended Koponen family of Boyarchenko and Levendorskii (2000) and the double exponential model of Kou (2002). Lewis (2001); Lipton (2002) show that all of the above-mentioned models allow for fast and easy computation of European option prices via one-dimensional Fourier transforms. A comprehensive reference on the subject of option-pricing in an exponential Lévy setting can be found in Boyarchenko and Levendorskii (2002), as well as Chapter 11 of Cont and Tankov (2004).

Despite their success, exponential Lévy models have some shortcomings. For example, because the log returns of any exponential Lévy process are independent and identically distributed, these models cannot exhibit volatility clustering (the tendency for volatility to rise sharply for short periods of time) or the leverage effect (the tendency for volatility to rise as asset prices fall); both of these phenomena are well-documented in literature. Lévy processes also exhibit constant jump intensities. However, a recent study of S&P500 index returns indicates that jump-intensities, like volatility, are stochastic (see Christoffersen, Jacobs, and Ornathanalai (2009)). To address these shortcomings, Carr and Wu (2004) add stochastic volatility (with correlation to the underlying) by stochastically time-changing a Lévy process. Notably, the models described in Carr and Wu (2004) maintain the analytic tractability that makes the class of exponential Lévy processes attractive.

In this paper, we address the need for volatility clustering, the leverage effect and stochastic jump intensity by modeling the returns process X by a Lévy-type process whose local characteristics $(\gamma_t, \sigma_t, \nu_t)$ are stochastic. We then use generalized Fourier transform techniques and singular perturbation methods to derive an explicit formula for the approximate price of a European-style derivative.

Much like geometric Brownian motion arises as special case of an exponential Lévy process, the class of fast mean-reverting and multiscale stochastic volatility models considered in Fouque, Papanicolaou, and Sircar (2000) and Fouque, Papanicolaou, Sircar, and Solna (2011) arise as a special subset of the class of models we consider. In fact, by removing jumps from our framework, one recovers the Fourier representation of the European option pricing formulas derived in Fouque, Papanicolaou, and Sircar (2000) and Fouque, Papanicolaou, Sircar, and Solna (2011).

The rest of this paper proceeds as follows. In section 2 we introduce a class of exponential Lévy-type models in which the volatility and jump-intensity are stochastically driven by a common fast-varying factor. In section 3 we derive an expression for the approximate price of a European option (Theorem 3.1) when

the underlying is described by the class of models introduced in section 2. We also quantify the accuracy of our pricing approximation (Theorem 3.2). In section 4, as an example of our framework, we extend the jump-diffusion model of Merton (1976) to include stochastic volatility and stochastic jump-intensity. We also compute (numerically) the implied volatility surface generated by this example. A calibration of the extended Merton model to S&P500 index options is performed in section 5. In section 6 we briefly describe how the class of models introduced in section 2 can be extended to allow for multiple driving factors of volatility and jump-intensity – one fast-varying factor and one slow-varying factor. Proofs are provided in an appendix.

2 Stochastic volatility and jump intensity Lévy-type processes

Let $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ be a probability space endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, which satisfies the usual conditions. Here, $\tilde{\mathbb{P}}$ is the risk-neutral pricing measure, which we assume is chosen by the market. The filtration \mathbb{F} represents the history of the market. For simplicity, we assume that the risk-free rate of interest is zero so that all non-dividend paying assets are $(\tilde{\mathbb{P}}, \mathbb{F})$ -martingales. All of our results can easily be extended to include constant or deterministic interest rates.

We consider a non-dividend paying asset S whose dynamics under $\tilde{\mathbb{P}}$ are described by the following Itô-Lévy stochastic differential equation (SDE)

$$(2.1) \quad \left. \begin{aligned} dS_t &= \sigma(Y_t)S_t d\tilde{W}_t + S_{t-} \int_{\mathbb{R}} (e^z - 1) d\tilde{N}_t(Y_t, dz), & S_0 &= \log x, \\ dY_t &= \left(\frac{1}{\varepsilon^2} \alpha(Y_t) - \frac{1}{\varepsilon} \Lambda(Y_t) \beta(Y_t) \right) dt + \frac{1}{\varepsilon} \beta(Y_t) d\tilde{B}_t, & Y_0 &= y, \\ d\langle \tilde{W}, \tilde{B} \rangle &= \rho dt, & |\rho| &\leq 1. \end{aligned} \right\} \quad (\text{under } \tilde{\mathbb{P}})$$

Here \tilde{W} and \tilde{B} are correlated Brownian motions and $\tilde{N}(Y, dz)$ is a compensated Poisson random measure

$$d\tilde{N}_t(Y_t, dz) = dN_t(Y_t, dz) - \zeta(Y_t)\nu(dz)dt, \quad \tilde{\mathbb{E}}[dN_t(Y_t, dz)|Y_t] = \zeta(Y_t)\nu(dz)dt,$$

We require that the measure ν satisfy

$$\int_{\mathbb{R}} \min(1, z^2) \nu(dz) < \infty, \quad \int_{|z| \geq 1} e^z \nu(dz) < \infty, \quad \text{and} \quad \int_{|z| \geq 1} |z| \nu(dz) < \infty.$$

The first integrability condition must be satisfied by all Lévy measures. The second integrability condition is needed to ensure $\tilde{\mathbb{E}}[S_t] < \infty$ for all $t \in \mathbb{R}^+$. The last integrability condition allows us to replace the indicator function that usually appears in the Lévy-Kintchine formula $\mathbb{I}_{\{|z| < 1\}}$ with the constant 1. Although we do not require it, a correlation of $\rho < 0$ between \tilde{W} and \tilde{B} would be consistent with the leverage effect (i.e. a drop in the value of S will usually be accompanied by an increase in volatility).

Note that both the volatility of S , given by $\sigma(Y)$, and the jump-intensity $\zeta(Y)\nu(dz)$, which controls the jumps of S , are driven by a common stochastic process Y . We require the existence of constants $\underline{\sigma}$, $\overline{\sigma}$ and $\overline{\zeta}$ such that $0 < \underline{\sigma} \leq \sigma(y) \leq \overline{\sigma} < \infty$ and $0 \leq \zeta(y) \leq \overline{\zeta} < \infty$ for all y in the state space of Y .

2.1 The driving process Y and model assumptions

The process Y is *fast-varying* in the following sense. Under the physical measure \mathbb{P} , the dynamics of Y are described by

$$dY_t = \frac{1}{\varepsilon^2} \alpha(Y_t) dt + \frac{1}{\varepsilon} \beta(Y_t) dB_t \quad \left. \vphantom{\frac{1}{\varepsilon^2}} \right\} \quad (\text{under } \mathbb{P})$$

where $B_t = \tilde{B}_t - \int_0^t \Lambda(Y_s) ds$ is a \mathbb{P} -Brownian motion. The generator of Y under \mathbb{P} is scaled by a factor of $1/\varepsilon^2$

$$\mathcal{A}_Y^\varepsilon = \frac{1}{\varepsilon^2} \left(\frac{1}{2} \beta^2(y) \partial_{yy}^2 + \alpha(y) \partial_y \right).$$

Thus, Y operates with an intrinsic time-scale ε^2 . We assume $\varepsilon^2 \ll 1$ so that the intrinsic time-scale of Y is small. Thus, Y is fast-varying. Let Y^1 be a diffusion process whose infinitesimal generator is \mathcal{A}_Y^1 (so that, in distribution, $Y_t = Y_{t/\varepsilon^2}^1$ under \mathbb{P}). We assume the following:

1. Under \mathbb{P} , the process Y^1 is ergodic and has a unique invariant distribution F_Y .
2. The smallest non-zero eigenvalue of $-\mathcal{A}_Y^1$ is strictly positive.
3. There exists a constant $C(k) < \infty$ such that

$$\sup_t \mathbb{E}[|Y_t^1|^k] \leq C(k).$$

4. The coefficients $\alpha(y)$ and $\beta(y)$ are smooth and at most polynomially growing in y .

Typical processes which satisfy assumptions 1, 2 and 3 above are

OU process :	$\alpha(y) = m - y,$	$\beta(y) = v,$	$F_Y \sim \text{Normal},$
CIR process :	$\alpha(y) = m - y,$	$\beta(y) = v\sqrt{y},$	$F_Y \sim \text{Gamma}.$

We further assume:

5. The functions $\sigma(y)$ and $\zeta(y)$ are smooth and that the solutions $\eta(y)$ and $\xi(y)$ to Poisson equations

$$(2.2) \quad \mathcal{A}_0 \eta = \sigma^2 - \langle \sigma^2 \rangle, \quad \mathcal{A}_0 \xi = \zeta - \langle \zeta \rangle, \quad \langle f \rangle := \int f(y) F_Y(dy).$$

6. The market price of volatility risk Λ is smooth and bounded above and below.

7. The existence of a unique strong solution to SDE (2.1).

As mentioned in the introduction, the class of models described by (2.1) are a natural extension of those considered in Fouque, Papanicolaou, and Sircar (2000). The key differences between the class of models we consider and those considered in Fouque, Papanicolaou, and Sircar (2000) are that (i) we allow for the underlying S to jump and (ii) we allow for the jump intensity to be stochastic.

The use of a fast-varying driving factor Y is warranted when the life of an option t is large compared to the characteristic time-scale ε^2 of the driving process (i.e., $t \gg \varepsilon^2$). A variogram analysis performed by Fouque, Papanicolaou, Sircar, and Sølna (2003a) indicates a fast-varying component of volatility with a characteristic time-scale of 1.7 days. Typically, then, the use of models with fast-varying driving processes is limited to options with expiries of greater than 17 days.

3 Option pricing

We wish to price a European-style option, which pays $H(S_t)$ at the maturity date $t > 0$. It will be convenient to introduce the returns process $X = \log S$. Using Itô's formula for Itô-Lévy processes (see Øksendal and Sulem (2005), Theorem 1.14) one derives

$$dX_t = \gamma(Y_t) dt + \sigma(Y_t) d\widetilde{W}_t + \int_{\mathbb{R}} z d\widetilde{N}_t(Y_t, dz), \quad X_0 = x,$$

where the drift $\gamma(Y_t)$ is given by

$$\gamma(Y_t) = -\frac{1}{2}\sigma^2(Y_t) - \zeta(Y_t) \int_{\mathbb{R}} (e^z - 1 - z)\nu(dz).$$

Using risk-neutral pricing, the value $u^\varepsilon(t, x, y)$ of the European option under consideration is

$$u^\varepsilon(t, x, y) = \widetilde{\mathbb{E}}_{x,y} [h(X_t)], \quad h(x) := H(e^x).$$

From the Kolmogorov backward equation we find that $u^\varepsilon(t, x, y)$ satisfies the following partial integro-differential equation (PIDE) and boundary condition (BC)

$$(3.1) \quad (-\partial_t + \mathcal{A}^\varepsilon) u^\varepsilon(t, x, y) = 0, \quad u^\varepsilon(0, x, y) = h(x).$$

Here, the partial integro-differential operator \mathcal{A}^ε is the generator of (X, Y) , which is defined on functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\mathcal{A}^\varepsilon f(x, y) := \lim_{t \searrow 0} \frac{1}{t} \left(\widetilde{\mathbb{E}}_{x,y} [f(X_t, Y_t)] - f(x, y) \right) \quad (\text{if the limit exists}).$$

If $f \in C_0^2(\mathbb{R}^2)$, the space of twice differentiable functions with compact support, then the above limit does exist (see Theorem 1.22 of Øksendal and Sulem (2005)) and \mathcal{A}^ε is given by

$$\begin{aligned}
(3.2) \quad \mathcal{A}^\varepsilon &= \frac{1}{\varepsilon^2} \mathcal{A}_0 + \frac{1}{\varepsilon} \mathcal{A}_1 + \mathcal{A}_2, \\
\mathcal{A}_0 &= \mathcal{A}_Y^1 = \frac{1}{2} \beta^2(y) \partial_{yy}^2 + \alpha(y) \partial_y, \\
\mathcal{A}_1 &= \rho \beta(y) \sigma(y) \partial_{xy}^2 - \Lambda(y) \beta(y) \partial_y, \\
(3.3) \quad \mathcal{A}_2 &= \gamma(y) \partial_x + \frac{1}{2} \sigma^2(y) \partial_{xx}^2 + \zeta(y) \int_{\mathbb{R}} (\theta_z - 1 - z \partial_x) \nu(dz).
\end{aligned}$$

Here, θ_z is the *shift operator*, which acts on functions $f \in \text{dom}(\mathcal{A}^\varepsilon)$ via $\theta_z f(x, y) := f(x + z, y)$. From now on, we *define* $\mathcal{A}^\varepsilon f(x, y)$ by expressions (3.2)-(3.3), for all f such that the partial derivatives and integrals exist at (x, y) .

To establish the accuracy of our pricing approximation (Theorem 3.2) we need for Cauchy problem (3.1) to have a unique point-wise classical solution. As such, we shall *assume* enough regularity of the coefficients $(\alpha, \beta, \Lambda, \sigma, \zeta)$ and the payoff function h so as to ensure that (3.1) has classical solution. This will be the case, for example, if the coefficients $\beta^2, \alpha, \beta\sigma, \Lambda\beta, \gamma, \sigma^2, \zeta$, the payoff function h and its first two derivatives are bounded. Note that the above conditions are sufficient, but not necessary, to ensure a classical solution. Please see Theorem B.1 of Appendix B.

3.1 Formal asymptotic analysis

For general $(\sigma, \zeta, \alpha, \beta, \Lambda)$ there is no analytic solution to (3.1). We notice, however, that terms containing ε in (3.1) are diverging in the small- ε limit, giving rise to a *singular* perturbation about the $\mathcal{O}(1)$ operator $(-\partial_t + \mathcal{A}_2)$. This special form suggests that we seek an asymptotic solution to PIDE (3.1). Thus, following Fouque, Papanicolaou, and Sircar (2000), we expand u^ε

$$(3.4) \quad u^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n u_n.$$

Our goal will be to find an approximation $u^\varepsilon = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)$. The choice of expanding in integer powers of ε is natural given the form of \mathcal{A}^ε . We will justify this expansion when we prove the accuracy of our pricing approximation in Theorem 3.2.

In the formal asymptotic analysis that follows, we insert expansion (3.4) into PIDE (3.1) and collect terms of like powers of ε , starting at the lowest order. The $\mathcal{O}(1/\varepsilon^2)$ and $\mathcal{O}(1/\varepsilon)$ terms are

$$\begin{aligned}
\mathcal{O}(1/\varepsilon^2) : \quad & 0 = \mathcal{A}_0 u_0, \\
\mathcal{O}(1/\varepsilon) : \quad & 0 = \mathcal{A}_1 u_0 + \mathcal{A}_0 u_1.
\end{aligned}$$

Noting that all terms in \mathcal{A}_0 and \mathcal{A}_1 take derivatives with respect to y , we choose $u_0 = u_0(t, x)$ and $u_1 = u_1(t, x)$. Continuing the asymptotic analysis, the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ terms are

$$(3.5) \quad \mathcal{O}(1) : \quad 0 = (-\partial_t + \mathcal{A}_2)u_0 + \mathcal{A}_0u_2,$$

$$(3.6) \quad \mathcal{O}(\varepsilon) : \quad 0 = (-\partial_t + \mathcal{A}_2)u_1 + \mathcal{A}_1u_2 + \mathcal{A}_0u_3,$$

where we have used the fact that $\mathcal{A}_1u_1 = 0$ in the $\mathcal{O}(1)$ equation. Equations (3.5) and (3.6) are equations of the form

$$(3.7) \quad \mathcal{A}_0u = \chi.$$

A solution u to (3.7) exists if and only if χ satisfies the *centering condition*

$$\langle \chi \rangle := \int \chi dF_Y = 0.$$

For the origin of the centering condition we refer the reader to equations (3.12)-(3.13) in Fouque, Papanicolaou, Sircar, and Solov'ev (2011) and the discussion therein. Applying the centering condition to (3.5) and (3.6) yields

$$(3.8) \quad \mathcal{O}(1) : \quad 0 = (-\partial_t + \langle \mathcal{A}_2 \rangle)u_0,$$

$$(3.9) \quad \mathcal{O}(\varepsilon) : \quad 0 = (-\partial_t + \langle \mathcal{A}_2 \rangle)u_1 + \langle \mathcal{A}_1u_2 \rangle.$$

Note, from and (3.5) and (3.8) we have

$$\begin{aligned} \mathcal{A}_0u_2 &= -(-\partial_t + \mathcal{A}_2)u_0 + (-\partial_t + \langle \mathcal{A}_2 \rangle)u_0 = -(\mathcal{A}_2 - \langle \mathcal{A}_2 \rangle)u_0 \\ &= -\frac{1}{2}(\sigma^2 - \langle \sigma^2 \rangle)(\partial_{xx}^2 - \partial_x)u_0 \\ &\quad - (\zeta - \langle \zeta \rangle) \left(-\int_{\mathbb{R}} (e^z - 1 - z)\nu(dz)\partial_x + \int_{\mathbb{R}} (\theta_z - 1 - z\partial_x)\nu(dz) \right) u_0 \\ (3.10) \quad &= -\mathcal{A}_0 \left(\frac{1}{2}\eta(\partial_{xx}^2 - \partial_x) - \xi \int_{\mathbb{R}} (e^z - 1 - z)\nu(dz)\partial_x + \xi \int_{\mathbb{R}} (\theta_z - 1 - z\partial_x)\nu(dz) \right) u_0, \end{aligned}$$

where we have used $\eta(y)$ and $\xi(y)$ as solutions to (2.2). Thus, from (3.9) and (3.10) we find

$$(3.11) \quad \mathcal{O}(\varepsilon) : \quad (-\partial_t + \langle \mathcal{A}_2 \rangle)u_1 = -\mathcal{B}u_0,$$

where the operator \mathcal{B} , which is defined to act on functions of x (and not y), is given by

$$\begin{aligned} \mathcal{B} &= \left\langle -\mathcal{A}_1 \left(\frac{1}{2}\eta(y)(\partial_{xx}^2 - \partial_x) - \xi \int_{\mathbb{R}} (e^z - 1 - z)\nu(dz)\partial_x + \xi \int_{\mathbb{R}} (\theta_z - 1 - z\partial_x)\nu(dz) \right) \right\rangle \\ (3.12) \quad &= V_3(\partial_{xxx}^3 - \partial_{xx}^2) + U_3 \left(-\int_{\mathbb{R}} (e^z - 1 - z)\nu(dz)\partial_{xx}^2 + \int_{\mathbb{R}} (\theta_z - 1 - z\partial_x)\partial_x\nu(dz) \right) \\ &\quad + V_2(\partial_{xx}^2 - \partial_x) + U_2 \left(-\int_{\mathbb{R}} (e^z - 1 - z)\nu(dz)\partial_x + \int_{\mathbb{R}} (\theta_z - 1 - z\partial_x)\nu(dz) \right), \end{aligned}$$

and the constants (V_3, U_3, V_2, U_2) are defined as

$$V_3 = -\frac{\rho}{2}\langle\beta\sigma\partial_y\eta\rangle, \quad U_3 = -\rho\langle\beta\sigma\partial_y\xi\rangle, \quad V_2 = \frac{1}{2}\langle\beta\Lambda\partial_y\eta\rangle, \quad U_2 = \langle\beta\Lambda\partial_y\xi\rangle.$$

Note that equality (3.12) holds only when \mathcal{B} is applied to functions that are independent of y . This is as far as we will take the asymptotic analysis. To review, we have found that $u_0(t, x)$ and $u_1(t, x)$ satisfy PIDEs (3.8) and (3.11) respectively. We also impose the following BCs

$$(3.13) \quad \mathcal{O}(1) : \quad u_0(0, x) = h(x),$$

$$(3.14) \quad \mathcal{O}(\varepsilon) : \quad u_1(0, x) = 0.$$

3.2 Explicit solution for $u_0(t, x)$ and $u_1(t, x)$

In order to find explicit formulas for $u_0(t, x)$ and $u_1(t, x)$, we note that the operator

$$(3.15) \quad \langle\mathcal{A}_2\rangle = \langle\gamma\rangle\partial_x + \frac{1}{2}\langle\sigma^2\rangle\partial_{xx}^2 + \langle\zeta\rangle\int_{\mathbb{R}}(\theta_z - 1 - z\partial_x)\nu(dz).$$

is the generator of a Lévy process with Lévy triplet $(\langle\gamma\rangle, \langle\sigma^2\rangle, \langle\zeta\rangle\nu)$. Thus, we may apply standard results from the classical theory of generalized Fourier transforms to obtain solutions to PIDEs (3.8) and (3.11).

Recall that the *generalized Fourier transform* of a function $f(x)$ is defined as

$$\widehat{f}(\lambda) := \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} f(x), \quad \lambda \in \{\mathbb{C} : |\widehat{f}(\lambda)| < \infty\}.$$

Assuming $\widehat{f}(\lambda)$ is analytic in an infinite strip in the complex plane $\Lambda = \{\lambda \in \mathbb{C} : \text{Im}(\lambda) \in (\lambda_-, \lambda_+)\}$, then one can recover the function $f(x)$ from $\widehat{f}(\lambda)$ by applying the inverse transform

$$f(x) = \int d\lambda_r \frac{1}{\sqrt{2\pi}} e^{i\lambda x} \widehat{f}(\lambda), \quad \lambda = \lambda_r + i\lambda_i, \quad \lambda_i \in (\lambda_-, \lambda_+),$$

where $\lambda_r, \lambda_i \in \mathbb{R}$. For a review of the generalized Fourier transforms as they related to Lévy processes, we refer the reader to any of the following: Boyarchenko and Levendorskii (2002); Lewis (2001); Lipton (2002).

Before presenting explicit solutions $u_0(t, x)$ and $u_1(t, x)$ we note that

$$\langle\mathcal{A}_2\rangle \frac{1}{\sqrt{2\pi}} e^{i\lambda x} = \phi_\lambda \frac{1}{\sqrt{2\pi}} e^{i\lambda x}, \quad \mathcal{B} \frac{1}{\sqrt{2\pi}} e^{i\lambda x} = B_\lambda \frac{1}{\sqrt{2\pi}} e^{i\lambda x}$$

where ϕ_λ and B_λ are given by

$$(3.16) \quad \phi_\lambda = i\langle\gamma\rangle\lambda - \frac{1}{2}\langle\sigma^2\rangle\lambda^2 + \langle\zeta\rangle\int_{\mathbb{R}}(e^{i\lambda z} - 1 - i\lambda z)\nu(dz),$$

$$(3.17) \quad \begin{aligned} B_\lambda = & V_3(-i\lambda^3 + \lambda^2) + U_3\left(\lambda^2\int_{\mathbb{R}}(e^z - 1 - z)\nu(dz) + i\lambda\int_{\mathbb{R}}(e^{i\lambda z} - 1 - i\lambda z)\nu(dz)\right) \\ & + V_2(-\lambda^2 - i\lambda) + U_2\left(-i\lambda\int_{\mathbb{R}}(e^z - 1 - z)\nu(dz) + \int_{\mathbb{R}}(e^{i\lambda z} - 1 - i\lambda z)\nu(dz)\right), \end{aligned}$$

which is valid for any $\lambda \in \mathbb{C}$ such that ϕ_λ is well-defined. Those who are familiar with Lévy processes will recognize ϕ_λ as the characteristic Lévy exponent corresponding to Lévy triplet $(\langle \gamma \rangle, \langle \sigma^2 \rangle, \langle \zeta \rangle \nu)$.

Theorem 3.1. *Assume that ϕ_λ , given by (3.16), is analytic in an infinite strip $\Lambda^\phi = \{\lambda \in \mathbb{C} : \text{Im}(\lambda) \in (\lambda_-^\phi, \lambda_+^\phi)\}$ of the complex plane, which contains the real axis. Assume $\widehat{h}(\lambda)$, the generalized Fourier transform of $h(x)$, is analytic in an infinite strip $\Lambda^h = \{\lambda \in \Lambda^\phi : \text{Im}(\lambda) \in (\lambda_-^h, \lambda_+^h)\}$. Let $\lambda = \lambda_r + i\lambda_i$ where $\lambda_r, \lambda_i \in \mathbb{R}$ and fix the imaginary component $\lambda_i \in (\lambda_-^h, \lambda_+^h)$. Then the solution $u_0(t, x)$ to PIDE (3.8) with BC (3.13) is*

$$u_0(t, x) = \int_{\mathbb{R}} d\lambda_r e^{t\phi_\lambda} \widehat{h}(\lambda) \frac{1}{\sqrt{2\pi}} e^{i\lambda x},$$

where ϕ_λ is given by (3.16), and the solution $u_1(t, x)$ to PIDE (3.11) with BC (3.14) is

$$u_1(t, x) = \int_{\mathbb{R}} d\lambda_r t e^{t\phi_\lambda} \widehat{h}(\lambda) B_\lambda \frac{1}{\sqrt{2\pi}} e^{i\lambda x}.$$

where B_λ is given by (3.17).

Proof. See appendix A. □

3.3 Accuracy of the pricing approximation

We have now derived an approximation $u^\varepsilon \approx u_0 + \varepsilon u_1$ for the price of any European option. However, this derivation relied on formal singular perturbation arguments. In what follows, we establish the accuracy of our pricing approximation. For our accuracy result, we shall need the following assumption:

- The payoff function $h(x)$ and all derivatives of $h(x)$ are smooth and bounded.

Obviously, many common derivatives – e.g., call and put options – do not fit this assumption. To prove the accuracy of our pricing approximation for calls and puts would require regularizing the option payoff and extending to unbounded functions, as is done in Fouque, Papanicolaou, Sircar, and Sølna (2003b). The regularization procedure is beyond the scope of this paper. As such, we limit our analysis to options with smooth and bounded payoffs. Our accuracy result is as follows:

Theorem 3.2. *For fixed (t, x, y) , there exists a constant C such that for any $\varepsilon \leq 1$ we have*

$$|u^\varepsilon - (u_0 + \varepsilon u_1)| \leq C \varepsilon^2.$$

Proof. See appendix C. □

Theorem 3.2 gives us information about how our pricing approximation behaves as $\varepsilon \rightarrow 0$. In practice ε is small, but fixed (it does not go to zero). Without knowing what the constant C is in theorem 3.2, it is

difficult to gauge exactly how good the pricing approximation is. As such, in the example provided in section 4, we will compare the approximate price $u_0 + \varepsilon u_1$ of a derivative-asset, calculated using the formulas in Theorem 3.1, to the full price u^ε , calculated via Monte Carlo simulation.

4 Example: Merton jump-diffusion with stochastic volatility and jump-intensity

In this section we provide one specific example within the class of models described in section 2. Specifically, we extend the jump-diffusion model of Merton (1976) to include stochastic volatility and jump-intensity. We refer to this class of models as the *Extended Merton* class or simply *ExtMerton*. In the Merton jump-diffusion model, jumps are log-normally distributed. Thus, we let the measure ν be given by

$$\nu(dz) = \frac{1}{\sqrt{2\pi s^2}} \exp\left(-\frac{(z-m)^2}{2s^2}\right) dz.$$

Under this specification, we have

$$\begin{aligned} \langle \gamma \rangle &= -\frac{1}{2} \langle \sigma^2 \rangle - \langle \zeta \rangle \left(e^{m+\frac{s^2}{2}} - m - 1 \right), \\ \phi_\lambda &= i \langle \gamma \rangle \lambda - \frac{1}{2} \langle \sigma^2 \rangle \lambda^2 + \langle \zeta \rangle \left(e^{i\lambda m - \frac{1}{2}s^2\lambda^2} - i\lambda m - 1 \right), \\ B_\lambda &= V_3 \left(-i\lambda^3 + \lambda^2 \right) + U_3 \left(\lambda^2 \left(e^{m+s^2/2} - 1 - m \right) + i\lambda \left(e^{i\lambda m - s^2\lambda^2/2} - 1 - i\lambda m \right) \right) \\ &\quad + V_2 \left(-\lambda^2 - i\lambda \right) + U_2 \left(-i\lambda \left(e^{m+s^2/2} - 1 - m \right) + \left(e^{i\lambda m - s^2\lambda^2/2} - 1 - i\lambda m \right) \right). \end{aligned}$$

For a European call option with payoff $h(X_t) = (e^{X_t} - e^k)^+$, the generalized Fourier transform of $h(x)$ is given by

$$\widehat{h}(\lambda) = \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} (e^x - e^k)^+ = \frac{-e^{k-ik\lambda}}{\sqrt{2\pi}(i\lambda + \lambda^2)}, \quad \text{Im}(\lambda) < -1.$$

The values of $(\langle \sigma^2 \rangle, \langle \zeta \rangle, V_3, U_3, V_2, U_2)$ depend on the particular choice of $\sigma(y)$ and $\zeta(y)$ as well as a specific choice for the Y process. In the numerical examples below we let $\alpha(y) = -y$, $\beta(y) = \beta$, and $\Lambda(y) = \Lambda$ so that

$$dY_t = \left(-\frac{1}{\varepsilon^2} Y_t - \frac{1}{\varepsilon} \Lambda \beta \right) dt + \frac{1}{\varepsilon} \beta d\widetilde{B}_t,$$

and we choose $\sigma(y) = ae^y$ and $\zeta(y) = be^y$. With these choices the invariant distribution of Y under the physical measure \mathbb{P} is normal $F_Y \sim \mathcal{N}(0, \frac{\beta^2}{2})$ and we can compute explicitly

$$\begin{aligned}\langle \sigma^2 \rangle &= a^2 e^{\beta^2}, & \langle \zeta \rangle &= be^{\frac{\beta^2}{4}}, \\ V_3 &= \frac{\rho}{\beta} a^3 e^{\frac{5\beta^2}{4}} (e^{\beta^2} - 1), & U_3 &= \frac{\rho}{\beta} 2ab \left(e^{\beta^2} - e^{\frac{\beta^2}{2}} \right), \\ V_2 &= -\beta \Lambda a^2 e^{\beta^2}, & U_2 &= -\beta \Lambda b e^{\frac{\beta^2}{4}}.\end{aligned}$$

The implied volatility I corresponding to a European call option with price u is defined implicitly though

$$u^{BS}(I) = u,$$

where $u^{BS}(I)$ is the price of the call option (with the same strike and maturity) as computed in the Black-Scholes framework assuming a volatility of I . In figure 1 we fix the time to maturity at $t = 1/10$ and we plot the Black-Scholes implied volatility smile induced by the approximate price of European calls $u_0 + \varepsilon u_1$ for different values of $\varepsilon = \{0.1, .033, 0.01\}$. For comparison, we also plot the implied volatility smile induced by the full price u^ε (computed using Monte Carlo simulation). As expected, as ε goes to zero, the implied volatility induced by the approximate price $u_0 + \varepsilon u_1$ converges to the implied volatility induced by the full price u^ε .

5 Calibration to S&P500 index options

In this section we calibrate ExtMerton jump-diffusion class discussed in section 4 to the implied volatility surface of S&P500 options. For comparison, we also calibrate the classical Merton model and the fast mean-reverting stochastic volatility (FMR-SV) class of models of Fouque, Papanicolaou, and Sircar (2000) to the same set of data.

In order to formulate the calibration procedure, it will be useful to introduce the following notation

$$\begin{aligned}\Phi &:= (\langle \sigma^2 \rangle, \langle \zeta \rangle, m, s, V_2^\varepsilon, V_3^\varepsilon, U_2^\varepsilon, U_3^\varepsilon), \\ \Theta &:= \{\Phi : \langle \sigma^2 \rangle > 0, \langle \zeta \rangle \geq 0, m \in \mathbb{R}, s \geq 0, (V_2^\varepsilon, V_3^\varepsilon, U_2^\varepsilon, U_3^\varepsilon) \in \mathbb{R}^4\},\end{aligned}$$

where we have defined $V_i^\varepsilon := \varepsilon V_i$ and $U_i^\varepsilon := \varepsilon U_i$. Note that the elements of Φ are the unobservable parameters needed to compute the approximate price of an option $u_0 + \varepsilon u_1$ in the ExtMerton framework, and Θ is the feasible state space of these parameters. Note also that we do not assume a specific value for ε , a specific volatility process Y , or specific functions: $\sigma(y)$ or $\zeta(y)$. In fact, this is one of the *main features* of the class of models considered in this paper. By assuming that the driving factor Y is fast-varying and

ergodic, specific choices for $(\varepsilon, Y, \sigma(y), \zeta(y))$ are *not needed* to compute the approximate price $u_0 + \varepsilon u_1$ of an option (or the corresponding implied volatility). For the purposes of calibration and pricing, the relevant information about $(\varepsilon, Y, \sigma(y), \zeta(y))$ is neatly contained in $\langle \sigma^2 \rangle$, $\langle \zeta \rangle$ and the four group parameters $\{V_i^\varepsilon, U_i^\varepsilon, i = 1, 2\}$.

Let $I(t, k)$ be the observed implied volatility of a European call option with time to maturity t and log strike $k = \log K$. Let $I^\varepsilon(t, k; \Phi)$ be the implied volatility of a European call option with the same maturity and strike as computed in the ExtMerton framework using parameters $\Phi \in \Theta$. We formulate the calibration problem for the ExtMerton class as a least squares optimization. That is, we seek Φ^* such that

$$(5.1) \quad \min_{\Phi \in \Theta} \sum_i (I(t_i, k_i) - I^\varepsilon(t_i, k_i; \Phi))^2 = \sum_i (I(t_i, k_i) - I^\varepsilon(t_i, k_j; \Phi^*))^2.$$

Here, the sum runs over all pairs (t_i, k_i) for which a call or put is liquidly traded and for which $t_i \geq 17$ days. The calibration procedures for the Merton model and the FMR-SV class are performed in a similar fashion by solving (5.1) for $\Phi \in \Theta^{Mer}$ and $\Phi \in \Theta^{FMR}$ respectively, where

$$\begin{aligned} \Theta^{Mer} &:= \{\Phi : \langle \sigma^2 \rangle > 0, \langle \zeta \rangle \geq 0, m \in \mathbb{R}, s \geq 0, (V_2^\varepsilon, V_3^\varepsilon, U_2^\varepsilon, U_3^\varepsilon) = 0\}, \\ \Theta^{FMR} &:= \{\Phi : \langle \sigma^2 \rangle > 0, (\langle \zeta \rangle, m, s) = 0, (V_2^\varepsilon, V_3^\varepsilon) \in \mathbb{R}^2, (U_2^\varepsilon, U_3^\varepsilon) = 0\}. \end{aligned}$$

Note that by requiring $(V_2^\varepsilon, V_3^\varepsilon, U_2^\varepsilon, U_3^\varepsilon) = 0$ in Θ^{Mer} the effects of stochastic volatility and stochastic jump intensity disappear, and the approximate option price in the ExtMerton class $u_0 + \varepsilon u_1$ reduces to the Merton price u_0 . Similarly, by requiring that $(\langle \zeta \rangle, m, s, U_2^\varepsilon, U_3^\varepsilon) = 0$ in Θ^{FMR} , the effect of the jumps disappears (the effects of stochastic volatility remain), and the approximate option price in the ExtMerton class $u_0 + \varepsilon u_1$ reduces to the price as computed in the FMR-SV class.

We perform the calibration procedure for all three frameworks (ExtMerton class, classical Merton model, and FMR-SV class) on two dates: October 2, 2006 and May 3, 2010. The dates were chosen to represent both the pre-crisis and post-crisis periods. To perform the calibration we use MATLAB's built-in non-linear least squares optimizer: `lsqnonlin`. The fit from the October 2, 2006 calibration (encompassing maturities of 47, 75, 166, 257, 355, 446 and 628 days) is plotted in figure 2. The shortest maturity is plotted on the top of the figure and the longest maturity at the bottom. For each plot, the units of the horizontal axis are *log-moneyness to maturity ratio*: $LMMR := \log(K/S_0 e^{rt})$. The vertical axis represents implied volatility.

The parameters obtained from the October 2, 2006 calibration procedure are listed below:

$$\begin{aligned}
\text{Extended Merton : } \quad & \langle \sigma^2 \rangle = 0.3605^2, & \langle \zeta \rangle = 0.0223, & m = -0.2226, & s = 0.4127, \\
& V_2^\varepsilon = -0.0156, & V_3^\varepsilon = 0.0001, & U_2^\varepsilon = 0.1304, & U_3^\varepsilon = -0.0481, \\
\text{Merton : } \quad & \langle \sigma^2 \rangle = 0.3605^2, & \langle \zeta \rangle = 0.1371, & m = -0.2750, & s = 0.1850, \\
\text{FMR-SV : } \quad & \langle \sigma^2 \rangle = 0.3605^2, & V_2 = -0.0767, & V_3 = -0.0050.
\end{aligned}$$

The fit from the May 3, 2010 calibration (encompassing maturities of 47, 75, 110, 138, 229, 320, 411 and 593 days) is plotted in figure 3. The parameters obtained from the May 3, 2010 calibration procedure are listed below:

$$\begin{aligned}
\text{Extended Merton : } \quad & \langle \sigma^2 \rangle = 0.1704^2, & \langle \zeta \rangle = 0.0287, & m = -0.3218, & s = 0.7974, \\
& V_2^\varepsilon = -0.0452, & V_3^\varepsilon = 0.0005, & U_2^\varepsilon = 0.0692, & U_3^\varepsilon = -0.0528, \\
\text{Merton : } \quad & \langle \sigma^2 \rangle = 0.3714^2, & \langle \zeta \rangle = 0.0896, & m = -0.5396, & s = 0.3974, \\
\text{FMR-SV : } \quad & \langle \sigma^2 \rangle = 0.3714^2, & V_2 = -0.0547, & V_3 = -0.0060.
\end{aligned}$$

In both cases, a visual inspection of figures 2 and 3, supports the use of the ExtMerton class over the Merton model and the FMR-SV class – especially for high strikes at the shortest maturities. The visual evidence is confirmed by the obtained residuals (i.e., the sum in equation (5.1) for the optimal Φ^* obtained in each calibration)

	Extended Merton	Merton	FMR-SV
October 2, 2006 :	0.0161,	0.0425	0.1306,
May 3, 2010 :	0.1184,	0.3930	1.0900.

While, for both dates tested, we find that the ExtMerton class provides a tighter fit to implied volatilities than both the Merton model and the FMR-SV class, it is apparent that the calibration for all three models performs poorly for the longest maturities. To obtain a tight fit across all maturities, it is likely that multiple driving factors – operating on different time scales – are required. We discuss this multiscale extension in the following section.

6 Extension to multiscale stochastic volatility and jump intensity

The results of this paper can be extended in a straightforward manner to include *multiscale* stochastic volatility and jump intensity. We briefly describe how this may be done. Our intent in this section is not to

be rigorous, but rather to give a flavor of the computations involved in this extension. To begin, we modify the dynamics of S slightly. Letting $S = e^X$ we have

$$\left. \begin{aligned} dX_t &= \gamma(Y_t, Z_t) dt + \sigma(Y_t, Z_t) d\widetilde{W}_t^x + \int_{\mathbb{R}} s d\widetilde{N}_t(Y_t, Z_t, ds), & X_0 &= x, \\ dY_t &= \left(\frac{1}{\varepsilon^2} \alpha(Y_t) - \frac{1}{\varepsilon} \Lambda(Y_t, Z_t) \beta(Y_t) \right) dt + \frac{1}{\varepsilon} \beta(Y_t) d\widetilde{W}_t^y, & Y_0 &= y, \\ dZ_t &= \left(\delta^2 c(Z_t) - \delta \Gamma(Y_t, Z_t) g(Z_t) \right) dt + \delta g(Z_t) d\widetilde{W}_t^z, & Z_0 &= z. \end{aligned} \right\} \quad (\text{under } \widetilde{\mathbb{P}})$$

Here, Z is a *slow-varying* factor, in the sense that its infinitesimal generator under \mathbb{P} is scaled by δ^2 , which is assumed to be a small parameter: $\delta^2 \ll 1$. The Brownian motions \widetilde{W}^x , \widetilde{W}^y , \widetilde{W}^z have correlations ρ_{xy} , ρ_{xz} and ρ_{yz} (which must be such that the covariance matrix is positive definite), the compensated Poisson random measure $\widetilde{N}(Y, Z, ds)$ satisfies

$$\begin{aligned} d\widetilde{N}_t(Y_t, Z_t, ds) &= dN_t(Y_t, Z_t, ds) - \zeta(Y_t, Z_t) \nu(ds) dt, \\ \widetilde{\mathbb{E}}[dN_t(Y_t, Z_t, ds) | Y_t, Z_t] &= \zeta(Y_t, Z_t) \nu(ds) dt, \end{aligned}$$

and the drift $\gamma(Y_t, Z_t)$ is given by

$$\gamma(Y_t, Z_t) = -\frac{1}{2} \sigma^2(Y_t, Z_t) - \zeta(Y_t, Z_t) \int_{\mathbb{R}} (e^s - 1 - s) \nu(ds).$$

Using risk-neutral pricing, the value $u^{\varepsilon, \delta}(t, x, y, z)$ of a European option in this setting is

$$u^{\varepsilon, \delta}(t, x, y, z) = \widetilde{\mathbb{E}}_{x, y, z} [h(X_t)], \quad h(x) := H(e^x).$$

From the Kolmogorov backward equation, the function $u^{\varepsilon, \delta}$ satisfies the following PIDE and BC

$$(6.1) \quad (-\partial_t + \mathcal{A}^{\varepsilon, \delta}) u^{\varepsilon, \delta} = 0, \quad u^{\varepsilon, \delta}(0, x, y, z) = h(x),$$

where the partial integro-differential operator $\mathcal{A}^{\varepsilon, \delta}$ is the generator of (X, Y, Z) . The operator $\mathcal{A}^{\varepsilon, \delta}$ has the following form

$$\mathcal{A}^{\varepsilon, \delta} = \frac{1}{\varepsilon^2} \mathcal{A}_0 + \frac{1}{\varepsilon} \mathcal{A}_1 + \mathcal{A}_2 + \frac{\delta}{\varepsilon} \mathcal{M}_3 + \delta \mathcal{M}_1 + \delta^2 \mathcal{M}_2.$$

Terms containing δ in (6.1) are small in the small- δ limit, giving rise to a *regular* perturbation. Thus, (6.1) has the form of a combined singular-regular perturbation about the $\mathcal{O}(1)$ operator $(-\partial_t + \mathcal{A}_2)$. Following Fouque, Papanicolaou, Sircar, and Solna (2011) we seek a solution $u^{\varepsilon, \delta}$ of the form

$$u^{\varepsilon, \delta} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon^n \delta^m u_{n, m}.$$

Our goal is to find an approximation $u^{\varepsilon, \delta} = u_{0,0} + \varepsilon u_{1,0} + \delta u_{0,1} + \mathcal{O}(\varepsilon^2 + \delta^2)$. A formal asymptotic analysis yields the following PIDEs for $u_{0,0}$, $u_{1,0}$ and $u_{0,1}$

$$\begin{aligned} \mathcal{O}(1) : \quad & (-\partial_t + \langle \mathcal{A}_2 \rangle) u_{0,0} = 0, & u_{0,0}(0, x, z) &= h(x), \\ \mathcal{O}(\varepsilon) : \quad & (-\partial_t + \langle \mathcal{A}_2 \rangle) u_{1,0} = -\mathcal{B} u_{0,0}, & u_{1,0}(0, x, z) &= 0, \\ \mathcal{O}(\delta) : \quad & (-\partial_t + \langle \mathcal{A}_2 \rangle) u_{0,1} = -\langle \mathcal{M}_1 \rangle u_{0,0}, & u_{0,1}(0, x, z) &= 0, \end{aligned}$$

where, as in section 3.1, the y -dependence has disappeared from $u_{0,0}$, $u_{1,0}$ and $u_{0,1}$. The operators $\langle \mathcal{A}_2 \rangle$, \mathcal{B} and $\langle \mathcal{M}_1 \rangle$ are given by

$$\begin{aligned} \langle \mathcal{A}_2 \rangle &= \langle \gamma(\cdot, z) \rangle \partial_x + \frac{1}{2} \langle \sigma^2(\cdot, z) \rangle \partial_{xx}^2 + \langle \zeta(\cdot, z) \rangle \int_{\mathbb{R}} (e^{s\partial_x} - 1 - s\partial_x) \nu(ds), \\ \mathcal{B} &= V_3(z) (\partial_{xxx}^3 - \partial_{xx}^2) + U_3(z) \left(-\int_{\mathbb{R}} (e^s - 1 - s) \nu(ds) \partial_{xx}^2 + \int_{\mathbb{R}} (\theta_s - 1 - s\partial_x) \partial_x \nu(ds) \right) \\ &\quad + V_2(z) (\partial_{xx}^2 - \partial_x) + U_2(z) \left(-\int_{\mathbb{R}} (e^s - 1 - s) \nu(ds) \partial_x + \int_{\mathbb{R}} (\theta_s - 1 - s\partial_x) \nu(ds) \right), \\ \langle \mathcal{M}_1 \rangle &= -g(z) \langle \Gamma(\cdot, z) \rangle \partial_z + g(z) \rho_{xz} \langle \sigma(\cdot, z) \rangle \partial_{xz}^2, \end{aligned}$$

where the z -dependent parameters $(V_3(z), U_3(z), V_2(z), U_2(z))$ are

$$\begin{aligned} V_3(z) &= -\frac{\rho_{xy}}{2} \langle \beta(\cdot) \sigma(\cdot, z) \partial_y \eta(\cdot, z) \rangle, & U_3(z) &= -\rho_{xy} \langle \beta(\cdot) \sigma(\cdot, z) \partial_y \xi(\cdot, z) \rangle, \\ V_2(z) &= \frac{1}{2} \langle \beta(\cdot) \Lambda(\cdot, z) \partial_y \eta(\cdot, z) \rangle, & U_2(z) &= \langle \beta(\cdot) \Lambda(\cdot, z) \partial_y \xi(\cdot, z) \rangle. \end{aligned}$$

The expressions for $u_{0,0}$ and $u_{1,0}$ are analogous to those given for u_0 and u_1 in Theorem 3.1. An expression for $u_{0,1}$ is obtained using the theory generalized Fourier transforms

$$\begin{aligned} \widehat{v}(s, \lambda, z) &:= \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} \langle \mathcal{M}_1 \rangle u_{0,0}(s, x, z) \\ u_{0,1}(t, x, z) &= \int_{\mathbb{R}} d\lambda_r \frac{1}{\sqrt{2\pi}} e^{i\lambda x} \int_0^t ds e^{(t-s)\phi_\lambda(z)} \widehat{v}(s, \lambda, z). \end{aligned}$$

where, as in Theorem 3.1, the Fourier variable satisfies $\lambda = \lambda_r + i\lambda_i$ where $\lambda_i \in (\lambda_-^h, \lambda_+^h)$. Note, care must be taken when computing $\langle \mathcal{M}_1 \rangle u_{0,0}$ as both terms in $\langle \mathcal{M}_1 \rangle$ contain the operator ∂_z and $u_{0,0}$ depends on z through both $\langle \sigma^2(\cdot, z) \rangle$ and $\langle \zeta(\cdot, z) \rangle$. A careful computation shows that $u_{0,1}$ is linear in the following four parameters

$$\begin{aligned} V_1(z) &= g(z) \rho_{xz} \langle \sigma(\cdot, z) \rangle \partial_z \langle \sigma^2(\cdot, z) \rangle, & V_0(z) &= -g(z) \langle \Gamma(\cdot, z) \rangle \partial_z \langle \sigma^2(\cdot, z) \rangle, \\ U_1(z) &= g(z) \rho_{xz} \langle \sigma(\cdot, z) \rangle \partial_z \langle \zeta(\cdot, z) \rangle, & U_0(z) &= -g(z) \langle \Gamma(\cdot, z) \rangle \partial_z \langle \zeta(\cdot, z) \rangle. \end{aligned}$$

Finally, the accuracy of the multiscale pricing approximation $u_{0,0} + \varepsilon u_{1,0} + \delta u_{0,1}$ is as follows: for fixed (t, x, y, z) there exists a constant C such that for any $\varepsilon \leq 1$, $\delta \leq 1$ we have

$$|u^{\varepsilon, \delta} - (u_{0,0} + \varepsilon u_{1,0} + \delta u_{0,1})| \leq C(\varepsilon^2 + \delta^2).$$

The proof of this error bound is analogous to the proof found in chapter 4 of Fouque, Papanicolaou, Sircar, and Solna (2011).

7 Conclusion

In this paper, we have introduced a class of exponential Lévy-type models in which the volatility and jump-intensity are driven stochastically by two factors – one fast-varying and one slow-varying. Using techniques from the theory of generalized Fourier transforms, singular and regular perturbation theory we have derived a general formula for the approximate price of a European-style derivative. Furthermore, we have quantified the accuracy of our pricing approximation both theoretically (see Theorem 3.2) and numerically (see figure 1). We hope this work motivates further research into exponential Lévy-type models. A possible extension of this paper would be to allow the jump *distribution* (rather than just the jump *intensity*) to vary stochastically in time.

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A Proof the Theorem 3.1

We wish to solve PIDE (3.8) with BC (3.13). Throughout this appendix, we always assume $\lambda = \lambda_r + i\lambda_i$ where $\lambda_i \in (\lambda_-^h, \lambda_+^h)$. Under the assumptions of Theorem 3.1 we have

$$\begin{aligned}
 \partial_t u_0(t, x) = \langle \mathcal{A}_2 \rangle u_0(t, x) &\quad \Rightarrow \quad \partial_t \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} u_0(t, x) = \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} \langle \mathcal{A}_2 \rangle u_0(t, x) \\
 &= \int_{\mathbb{R}} dx u_0(t, x) \langle \mathcal{A}_2 \rangle^* \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} \\
 &= \phi_\lambda \int_{\mathbb{R}} dx u_0(t, x) \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} \\
 \text{(A.1)} \quad &\Rightarrow \quad \partial_t \widehat{u}_0(t, \lambda) = \phi_\lambda \widehat{u}_0(t, \lambda), \\
 \text{(A.2)} \quad u_0(0, x) = h(x) &\quad \Rightarrow \quad \widehat{u}_0(0, \lambda) = \widehat{h}(\lambda).
 \end{aligned}$$

Note that $\langle \mathcal{A}_2 \rangle^*$, the formal adjoint of $\langle \mathcal{A}_2 \rangle$, is given by making the replacements $\partial_x \rightarrow -\partial_x$ and $\theta_z \rightarrow \theta_{-z}$ in (3.15). Using (A.1) and (A.2) one deduces

$$\widehat{u}_0(t, \lambda) = e^{t\phi_\lambda} \widehat{h}(\lambda) \quad \Rightarrow \quad u_0(t, x) = \int_{\mathbb{R}} d\lambda_r e^{t\phi_\lambda} \widehat{h}(\lambda) \frac{1}{\sqrt{2\pi}} e^{i\lambda x}.$$

To solve PIDE (3.11) with BC (3.14) we first note that

$$\mathcal{B}u_0(t, x) = \int_{\mathbb{R}} d\mu_r e^{t\phi_\mu} \widehat{h}(\mu) \mathcal{B} \frac{1}{\sqrt{2\pi}} e^{i\mu x} = \int_{\mathbb{R}} d\mu_r e^{t\phi_\mu} \widehat{h}(\mu) B_\mu \frac{1}{\sqrt{2\pi}} e^{i\mu x},$$

where $\mu = \mu_r + i\lambda_i$. Thus,

$$\begin{aligned}
 \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} \mathcal{B}u_0(t, x) &= \int_{\mathbb{R}} d\mu_r e^{t\phi_\mu} \widehat{h}(\mu) B_\mu \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi}} e^{-i\lambda x} \frac{1}{\sqrt{2\pi}} e^{i\mu x} \\
 &= \int_{\mathbb{R}} d\mu_r e^{t\phi_\mu} \widehat{h}(\mu) B_\mu \delta(\mu_r - \lambda_r) \\
 &= e^{t\phi_\lambda} \widehat{h}(\lambda) B_\lambda,
 \end{aligned}$$

where we have used $\frac{1}{2\pi} \int_{\mathbb{R}} dx e^{-i(\lambda_r - \mu_r)x} = \delta(\lambda_r - \mu_r)$. Therefore multiplying PIDE (3.11) and BC (3.14) by $\frac{1}{\sqrt{2\pi}} e^{-i\lambda x}$ and integrating with respect to x one finds

$$(-\partial_t + \phi_\lambda) \widehat{u}_1(t, \lambda) = -e^{t\phi_\lambda} \widehat{h}(\lambda) B_\lambda \quad \text{and} \quad \widehat{u}_1(0, \lambda) = 0.$$

Solving for $\widehat{u}_1(t, \lambda)$ we have

$$\widehat{u}_1(t, \lambda) = te^{t\phi_\lambda} \widehat{h}(\lambda) B_\lambda \quad \Rightarrow \quad u_1(t, x) = \int_{\mathbb{R}} d\lambda_r te^{t\phi_\lambda} \widehat{h}(\lambda) B_\lambda \frac{1}{\sqrt{2\pi}} e^{i\lambda x},$$

which completes the proof.

B Existence of classical solutions Cauchy problems

A classical existence and uniqueness result for second order integro-differential Cauchy problems with regular coefficients is established in Theorem 3.2, Chapter 3 of Bensoussan and Lions (1984). Below, we present the results of this Theorem in a form which is convenient for our purposes. We introduce the following function spaces:

- $W^{2,p}(\mathbb{R}^n)$ is the space of functions u such that $u \in L^p(\mathbb{R}^n)$, $\partial_i u \in L^p(\mathbb{R}^n)$ for all i and $\partial_{ij}^2 u \in L^p(\mathbb{R}^n)$ for all i, j .
- $L^p(0, T; W^{2,p}(\mathbb{R}^n))$ is the space of functions u such that $\left\| \|u\|_{W^{2,p}(\mathbb{R}^n)} \right\|_{L^p((0,T))} < \infty$.
- $L^p(0, T; L^p(\mathbb{R}^n))$ is the space of functions u such that $\left\| \|u\|_{L^p(\mathbb{R}^n)} \right\|_{L^p((0,T))} < \infty$.
- $\mathcal{W}^{2,1,p}((0, T) \times \mathbb{R}^n)$ is the space of functions $u \in L^p(0, T; W^{2,p}(\mathbb{R}^n))$ such that $\partial_t u \in L^p(0, T; L^p(\mathbb{R}^n))$.

The above norms are the usual $L^p(\Omega)$ and $W^{k,p}(\Omega)$ norms (below, α is a multi-index)

$$\|u\|_{L^p(\Omega)} = \left(\int_{\Omega} |u(x)|^p dx \right)^{1/p}, \quad \|u\|_{W^{2,p}} = \left(\sum_{|\alpha| \leq 2} \|\partial^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

Theorem B.1. *Let \mathcal{A} be given by*

$$\mathcal{A} = \sum_{i,j=1}^n a_{ij}(x) \partial_{ij} + \sum_{i=1}^n b_i(x) \partial_i + c(x) \int_{\mathbb{R}^n} \nu(dz) (\theta_z - 1 - z \cdot \nabla),$$

where $x \in \mathbb{R}^n$. Assume

$$a_{ij}, b_i, c \in L^\infty(\mathbb{R}^n) \quad \text{and} \quad \sum_{ij} a_{ij} x_i x_j \geq \alpha |x|^2, \forall x \in \mathbb{R}^n, \alpha > 0.$$

Then for $g \in L^2(0, T; L^p(\mathbb{R}^n))$ and $h \in W^{2,p}(\mathbb{R}^n)$, $2 \leq p < \infty$, the problem

$$(-\partial_t + \mathcal{A})u = g, \quad u(0, x) = h(x), \quad u \in \mathcal{W}^{2,1,p}((0, T) \times \mathbb{R}^n),$$

has a unique classical solution. Furthermore, if g, h are bounded, then u is bounded.

C Proof of accuracy

Before establishing our main accuracy result – Theorem 3.2 – we shall need the following lemma.

Lemma C.1. *Suppose $J(y)$ is at most polynomially growing. Then, under the assumptions of section 2.1, for every y and $s < t$, there exists a positive constant $C < \infty$ such that for any $\varepsilon \leq 1$, we have the following inequality*

$$\tilde{\mathbb{E}}_y [|J(Y_s)|] \leq C.$$

Proof of Lemma C.1. See Lemma 4.9 of Fouque, Papanicolaou, Sircar, and Solna (2011). \square

We are now in a position to prove Theorem 3.2. We begin by defining a remainder term R^ε by

$$u^\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + R^\varepsilon.$$

From the equations satisfied by u_0, u_1, u_2 and u_3 in section 3.1, one derives

$$(C.1) \quad (-\partial_t + \mathcal{A}^\varepsilon) R^\varepsilon = -\varepsilon^2 F^\varepsilon, \quad R^\varepsilon(0, x, y) = -\varepsilon^2 G^\varepsilon(x, y).$$

where we have defined

$$F^\varepsilon(t, x, y) := (\mathcal{A}_1 u_3 + \mathcal{A}_2 u_2) + \varepsilon \mathcal{A}_2 u_3, \quad G^\varepsilon(x, y) := u_2(0, x, y) + \varepsilon u_3(0, x, y).$$

Now, fix some finite T . For $t \in [0, T]$ we note that F^ε and G^ε are smooth functions of t, x, y that are, for $\varepsilon \leq 1$, bounded by smooth functions of t, x, y independent of ε , uniformly bounded in t, x and at most polynomially growing in y (see Fouque, Papanicolaou, Sircar, and Solna (2011), p. 143). This assertion can be deduced as follows. First, since h is smooth and bounded (by assumption), then by Theorem B.1, the function u_0 is bounded. Likewise, since ∂_x^n commutes with $\langle \mathcal{A}_2 \rangle$, and since all derivatives $\partial_x^n h$ are bounded (by assumption), then by Theorem B.1, one deduces that $\partial_x^n u_0$ is bounded for every n . Now, we note that u_2 and u_3 are completely characterized by u_0 and u_1 through Poisson equations (3.5) and (3.6). From (3.10) one finds that u_2 is given by

$$u_2 = -\left(\frac{1}{2}\eta(\partial_{xx}^2 - \partial_x) - \xi \int_{\mathbb{R}} (e^z - 1 - z)\nu(dz)\partial_x + \xi \int_{\mathbb{R}} (\theta_z - 1 - z\partial_x)\nu(dz)\right)u_0 + D$$

where $D(t, x)$ is an arbitrary function, which we choose to be zero (the choice of D does not effect our accuracy result). Because all derivatives $\partial_x^n u_0$ are bounded, one deduces that u_2 is uniformly bounded in t, x . The y -dependence in u_2 enters only through the solutions η and ξ of Poisson equations (2.2), which we assumed are at most polynomially growing in y . Likewise, derivatives $\partial_x^n u_2$ and $\partial_y^n u_2$ are uniformly bounded in t, x and at most polynomially growing in y . A similar argument shows that u_3 and its derivatives $\partial_x^n u_3$ and $\partial_y^n u_3$ are uniformly bounded in t, x and at most polynomially growing in y . Now, note that F^ε also has y -dependence through the operators \mathcal{A}_1 and \mathcal{A}_2 . These operators depend on y through the coefficients σ, Λ, ζ , which are all bounded by assumption, and through β , which is at most polynomially growing in y (again, by assumption). Thus, as claimed, F^ε and G^ε (as well as u_2 and u_3) are uniformly bounded in t, x and at most polynomially growing in y . And, for $\varepsilon \leq 1$, the functions F^ε and G^ε are bounded by functions of t, x, y , which, independent of ε , are uniformly bounded in t, x and at most polynomially growing in y .

Now, note that the solution R^ε of (C.1) has the following Feynman-Kac representation

$$(C.2) \quad R^\varepsilon(t, x, y) = \varepsilon^2 \tilde{\mathbb{E}}_{x,y} \left[-G^\varepsilon(X_t, Y_t) + \int_0^t F^\varepsilon(s, X_s, Y_s) ds \right].$$

The validity of the Feynman-Kac representation is guaranteed because (i) the expectation (C.2) is clearly finite (due to the uniform bound in $s < t$ and x for F^ε and in x for G^ε , the polynomial growth bound in y for F^ε and G^ε , and the existence of moments $\tilde{\mathbb{E}}[|Y|^k]$ of any order), and (ii) the function $R^\varepsilon := u^\varepsilon - (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3)$ is smooth by construction.

Using representation (C.2) for R^ε and Lemma C.1 there exists a constant $C_1 < \infty$ such that

$$|R^\varepsilon| \leq \varepsilon^2 C_1.$$

Finally, using the the bounds on u_2 and u_3 we have

$$|u^{\varepsilon,\delta} - (u_0 + \varepsilon u_1)| \leq |R^\varepsilon| + |\varepsilon^2 u_2 + \varepsilon^3 u_3| \leq \varepsilon^2 C_1 + \varepsilon^2 |u_2 + \varepsilon u_3| \leq \varepsilon^2 C,$$

for some constant $C < \infty$. This concludes the proof of accuracy.

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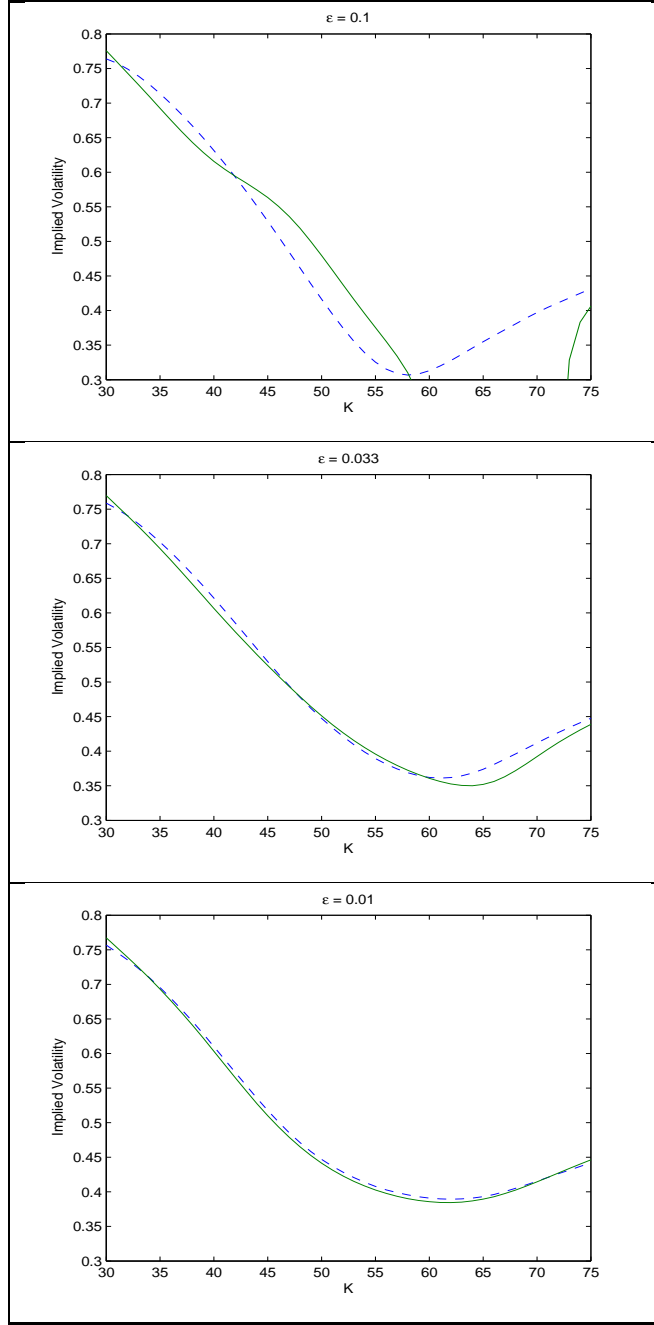


Figure 1: Using the model described in section 4, we plot the implied volatility induced by the price of European call option as a function of the strike price K . In each plot, the dashed blue line corresponds to the implied volatility induced by the full price u^ε (computed via Monte Carlo simulation) and the solid green line corresponds to the implied volatility induced by our approximation $u_0 + \varepsilon u_1$. For all plots we use the following parameter values: $t = 1/10$, $e^x = 50$, $m = -0.2$, $s = 0.2$, $\rho = -0.7$, $a = 0.2$, $b = 1.5$, $\beta = 1.0$ and $\Lambda = 0.25$.

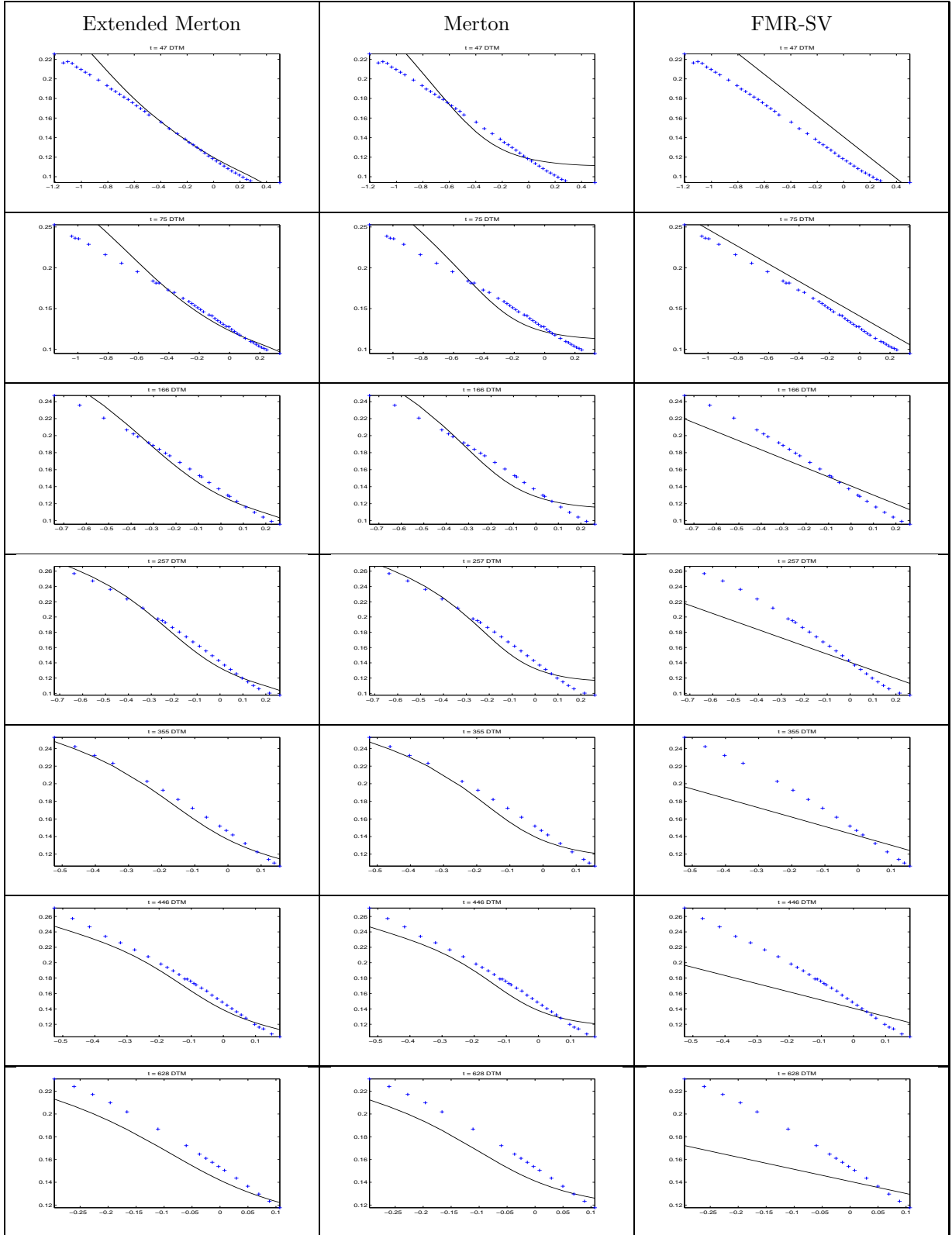


Figure 2: Implied volatility fit to S&P500 options from October 2, 2006.

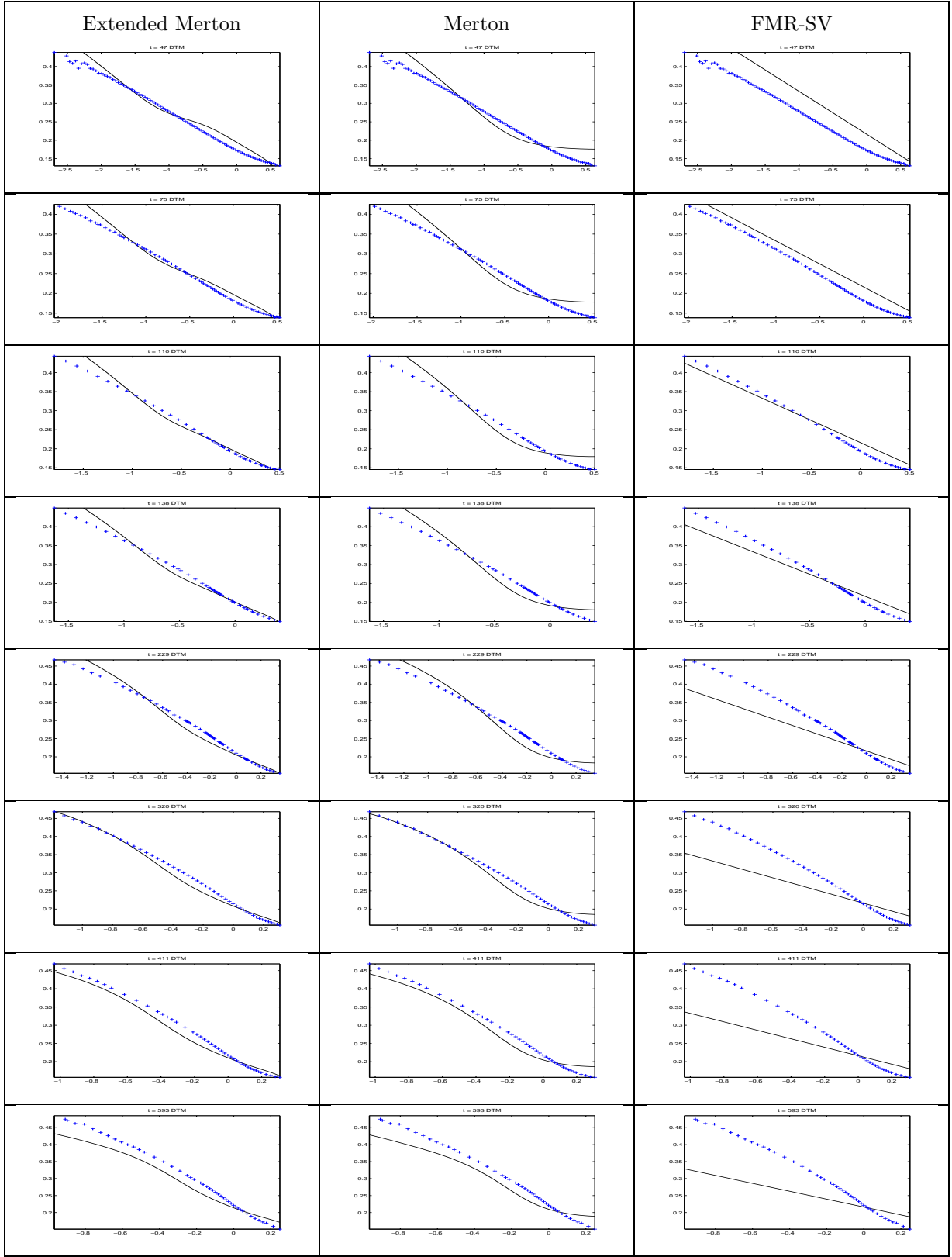


Figure 3: Implied volatility fit to S&P500 options from May 3, 2010.